Fixing the Frame in VLBI

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October 30, 1998

Abstract

In this note I derive the invariances of the VLBI delay. Each invariance is associated with a degeneracy of the normal equations. I describe the constraints we currently use to remove these degeneracies.

1   Introduction

Very long baseline interferometry (VLBI) is fundamentally a differential technique: It measures the difference in arrival time of an extragalactic signal at two or more antennas. Ignoring various modeling and nuisance effects the delay takes the form:

$$\tau_{ij,a} = \widehat{s_a}(t) \cdot (\overrightarrow{r_i}(t) - \overrightarrow{r_j}(t)) \tag{1}$$

where I have included explicit time dependence for both the source and the stations. The delay is sensitive to the location of both the receiving antennas and the sources. There are two natural reference frames in VLBI: the Celestial Reference Frame, (CRF), defined by the position of a set of quasars, and the Terrestial Reference Frame (TRF) defined by station locations on the Earth. The largest source of time variation of the delay comes from the change in orientation of the Earth with respect to the quasars. Most of this change is well predicted, for example, the strictly diurnal rotation of the Earth, and most of the nutation precession effects. A small part of the change is truly stochastic, or not yet well modeled. Because of this it us useful to rewrite Eq. (1) as:

$$\tau_{ij,a} = \widehat{s_a}(t) \cdot U(\omega(t)) U(\Omega_0(t)) (\overrightarrow{r_i}(t) - \overrightarrow{r_j}(t)) \tag{2}$$

Here $U(\Omega_0(t))$ and $U(\omega(t))$ are both rotation matrices. $U(\Omega_0(t))$ is the well predicted part of the rotation matrix, while $U(\omega(t))$ is the unmodeled part which must be supplied externally, or estimated from the data. I will call this latter unmodeled part the Earth Orientation Parameters (EOP). The advantage of this form for the delay is that with a suitable choice of rotation matrices the sources and station positions are approximately constant.

One of the goals of VLBI is to measure source positions, station positions and velocities, and the EOP matrix. From the form of the delay it is obvious that there are some transformations of these quantities that leave the delay unchanged, for example shifting all stations by a constant amount. We call such a transformation a "symmetry" or an invariance of the delay. Such a symmetry leads to a degeneracy in the normal equations, which must be removed by constraints or some other means.

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In section 2 we study the general effect of symmetries on normal equations, and prove a variety of interesting and important facts. In section 3 we study the symmteries associated with the VLBI delay, and show that there are 15 independent symmetries. This means that we must impose a minimum of 15 independent constraints to invert the normal equations. In section 4 we review the old style constraints used in VLBI. In the following sections we examine the new style constraints–the "No Net" constraints. This is followed by some concluding remarks. An appendix describes how to impose the constraints.

2 Symmetries, Invariances, Degeneracies and the Normal Equations

In this section we study the effect of symmetries on the normal equations. A symmetry is a transformation of the estimated parameters which leaves the observable invariant. We prove the following general results, some obvious, and some not so obvious:

1. Each symmetry leaves the normal equations invariant.

2. Each symmetry is associated with a degeneracy of the normal equations. That is, for each symmetry there is an eigenvector of the normal matrix with 0 eigenvalue.

3. Conversely, for each null-eigenvector (eigenvector with 0-eigenvalue), there is an associated symetry of the observable.

4. The number of independent symmtries is equal to the dimension of the null space of the normal matrix.

5. Each symmtry is assocated with a one parameter solution to the normal equations.

We start by reviewing and defining the normal equation formalism. Suppose that we have some observable τ which is a function of the n parameters A_i:

$$\tau = \tau(A_i)$$

Suppose we have some initial values $A_{i,0}$, for the parameters which is "close" to the final values. Since we know the functional form of τ we can expand it in a Taylor series around these values:

$$\tau(A_i) = \tau(A_{i,0}) + \sum_{i=1}^{n} \frac{\partial \tau}{\partial A_i} \Delta A_i + O(\Delta A_i^2)$$

If ΔA_i are sufficiently small, then the last term can be neglected. Our goal is to determine the A_i from a set of *nobs* measurements which we label by the subscript a. By assumption, we can write any observation as:

$$
\begin{aligned}
\tau_{a,obs} &= \tau_a(A_{i,0}) + \sum_{i=1}^{n} \frac{\partial \tau_a}{\partial A_i} \Delta A_i + \varepsilon_a \\
&= \tau_{a,calc} + \sum_{i=1}^{n} \Delta A_i f_i(a) + \varepsilon_a
\end{aligned}
$$

Where ε_a is the observational noise and for brevity in what follows we have made the replacements:

$$
\begin{aligned}
\tau_{a,calc} &\equiv \tau_a(A_{i,0}) \\
f_i(a) &= \frac{\partial \tau_a}{\partial A_i}
\end{aligned}
$$

The least squares technique starts by forming the sum over all observations of the difference between the measured and a priori values for the observable:

$$
\chi^2 = \sum_a \left[\tau_{a,obs} - \tau_{a,calc} - \sum_{i=1}^n \Delta A_i f_i(a) \right]^2 \frac{1}{\sigma_a^2}
$$

Our goal is to find the ΔA_i which minimizes this. This is easily done by differentiating the Right Hand Side (RHS) with respect to the B_i, and setting the result to zero. Sparing the details, we obtain:

$$
\sum_{j=1}^n \left[\sum_a \frac{1}{\sigma_a^2} f_i(a) f_j(a) \right] \Delta A_j = \sum_a \left(\tau_{a,obs} - \tau_{a,calc} \right) f_i(a)
$$

which can be written in the compact matrix notation:

$$
N \Delta A = B \tag{3}
$$

where N is an $n \times n$ matrix, and A and B are n dimensional column vectors. Explicitly:

$$
\begin{aligned}
N_{ij} &\equiv \sum_a \frac{1}{\sigma_a^2} f_i(a) f_j(a) \\
B_i &= \sum_a \left(\tau_{a,obs} - \tau_{a,calc} \right) f_i(a)
\end{aligned}
$$

We will refer to Eq. (3) as the normal equations, and any solution A of this equation as a solution of the normal equations. The formal solution to this equation is:

$$
\Delta A = N^{-1} B
$$

However, for this to work, the normal matrix N must be invertible. For notational simplicity, in what follows we will drop the Δ on ΔA.

With this formalism out of the way, we now turn to the effect of symmetries of the observable on the normal equation. First we need to define symmetry. Suppose we have some linear transformation of the A's which leaves the observable invariant. That is, we have an $n \times n$ matrix S (the linear transformation) such that

$$
A' = SA
$$

and

$$
\sum_{i=1}^n A'_i f_i(a) = \sum_{i=1}^n A_i f_i(a)
$$

for all observations a. We call such a transformation S a symmetry of the observable. We now prove some interesting facts.

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1.) If S is a symmetry of the observable, and A is a solution to the normal equations, then $A' = SA$ is a solution to the normal equations. **Proof.** By assumption we have

$$NA = B$$

We want to show that

$$NA' = B$$

Now, if we consider the Left Hand Side of this, we have

$$
\begin{aligned}
NA' &= \sum_{j=1}^{n} \left[\sum_{a} \frac{1}{\sigma_a^2} f_i(a) f_j(a) \right] A'_j \\
&= \sum_{a} \frac{1}{\sigma_a^2} f_i(a) \sum_{j=1}^{n} f_j(a) A'_j \\
&= \sum_{a} \frac{1}{\sigma_a^2} f_i(a) \sum_{j=1}^{n} f_j(a) A_j \\
&= \sum_{j=1}^{n} \left[\sum_{a} \frac{1}{\sigma_a^2} f_i(a) f_j(a) \right] A_j = NA
\end{aligned}
$$

In going from the first to the second line we have just reversed the order of summation. The step from the second to the third line uses the definition of symmetry, and in going to the last line, we again reverse the order of summation.

2.) Every continuous symmetry of the normal equations leads to a one parameter family of solutions of the normal equations. A continuous symmetry is a transformation $S(s)$ which leaves the observable invariant. **Proof.** This is an obvious extension of the previous result.

3.) Every continuous differentiable symmetry has a null-eigenvector associated with it. **Proof.** By assumption, the transformation can be expanded in terms of a Taylor series:

$$S(s) = S(s_0) + \Delta s \frac{\partial S(s_0)}{\partial s}$$

Now, it follows that both $S(s)$ and $S(s_0)$ are symmetries. Hence we have

$$
\begin{aligned}
NS(s)A &= N \left[S(s_0) + \Delta s \frac{\partial S(s_0)}{\partial s} \right] A \\
NA &= NA + N \left[\Delta s \frac{\partial S(s_0)}{\partial s} \right] A
\end{aligned}
$$

where in going from the first to the second line I have used the fact that $NS(s)A = NA$ for an arbitrary s. From this equation it follows directly that:

$$N \left\{ \frac{\partial S(s_0)}{\partial s} A \right\} = 0$$

hence the term in $\{\}$ is a null eigenvector of the normal equation.

4.) Let V be a null eigenvector of the normal equations

$$NV = 0$$

Then there is associated a one parameter family of solutions to the normal equations. In particular, if A is a solution to the normal equations, then so $A + \alpha V$ for arbitrary α. The proof is immediate.

5.) Let V be a null eigenvector of the normal equations. Then there is associated with it a one parameter family of transformations which leaves the delay invariant, that is, it is associated with a symmetry of the system. Consider the transformation

$$A \rightarrow A + \alpha V$$

This is a one parameter transformation. I will show that this is a symmetry of the observable, i.e., that it leaves the observable invariant. This is equivalent to the statement that

$$\sum_{j=1}^{n} f_j(a) A_j = \sum_{j=1}^{n} f_j(a)(A_j + \alpha V_j)$$

or

$$\sum_{j=1}^{n} f_j(a) V_j = 0$$

Proof: Consider the sum

$$\sum_a \frac{1}{\sigma_a^2} \left(\sum_{i=1}^{n} f_i(a) V_i \right)^2$$

which is the sum of a series of squares, and hence will vanish only if each of the squares vanishes. So if we can prove that this vanishes, then we are done. However this can be rewritten as:

$$\sum_a \frac{1}{\sigma_a^2} \left(\sum_{i=1}^{n} f_i(a) V_i \right) \left(\sum_{j=1}^{n} f_j(a) V_j \right) = VNV$$

which vanishes since $NV = 0$ by assumption. Hence we have proved what we wanted to prove.

In summary, in this section we have shown a deep relationship between 1.) continuous symmetries of the observable; 2.) degeneracies of the normal equations; and 3.) families of solution to the normal equations. In fact, all three of these are different ways of talking about the same thing.

3   Adding Constraints to Remove Degeneracies

One approach to remove the degeneracies of the normal equations is to add constraints to the system. This is most conveniently done by modifying the normal equation as follows:

$$N' = N + \sum_r \alpha_r U_r \otimes U_r$$

Here is the original normal equations, and the U_r are the constraint vectors, and the α_r are constants which determine how strong the constraint is. The U_r don't have to lie in the null space, they just need to have non-vanishing components in the null space. Let e_a be a basis for the null space, i.e., we can expand every vector in the null space in terms of the e_a. Then the modified normal equations will be invertible if

$$M_{ab} = \sum_r (e_a \cdot U_r)(e_b \cdot U_r)$$

is invertible. Another way of saying this is that the U_r span the null space. If the number of constraint vectors is equal to the dimension of the null-space, the set of constraints is said to be minimal. In this particular case you can prove that the solution to the normal equations is independent of the α_r. If the number of constraint vectors is larger than the dimension of the null-space, we have overconstrained the system. Even in the case of minimal constraints there is a lot of freedom in choosing the constraint vectors, since the only requirement is that they span the null space.

We now prove that if one the U_r has a component which lies in the null space, then

$$U_r \cdot A = 0$$

i.e., the solution is orthogonal to the U_r. **Proof.** Let V be a null eigenvector. Then it follows that

$$
\begin{aligned}
V \cdot B &= VNA \\
&= 0
\end{aligned}
$$

Hence B is orthogonal to the null space. (This is an elementary fact of linear algebra, but is worth repeating here.) By assumption, U_r has a component in the null space. This means there exists a vector e_a in the null space such that

$$U_r \cdot e_a \neq 0$$

Now, consider the modified normal equation. If we multiply this equation on both sides by e_a:

$$e_a \left(N + \sum_r \alpha_r U_r \otimes U_r \right) A = e_a B$$

We just showed that the right hand side of this is zero, hence the left hand side must also be zero. Since $e_a N = 0$ it follows that

$$(e_a \cdot U_r)(U_r \cdot A) = 0$$

Since $(e_a \cdot U_r)$ is non vanishing it follows that

$$U_r \cdot A = 0$$

The importance of this result is that this shows how the constraints act on the system. They force A to have a vanishing projection in some direction in parameter space. For exmaple, in VLBI if you want to constrain the Westford X-adjustment to 0, then the constraint vector would consist of a vector with a single non-vanishing component corresponding to the Westford X-adjustment.

4 Symmetries of the VLBI Delay

The VLBI delay takes the form

$$\tau_{ij,a} = \widehat{s_a}(t) \cdot U(\omega(t))(\overrightarrow{r_i}(t) - \overrightarrow{r_j}(t))$$

where I have ignored the large diurnal rotation which complicates the following discussion without substantially changing the results. Roughly speaking their are two kinds of symmetries associated with the delay. The first kind is due to symmetries of the baseline vector $\overrightarrow{r_i}(t) - \overrightarrow{r_j}(t)$, while the

second is related to transformations which leave the orientation of the TRF with respect to the CRF alone.

It is customary to model the station positions as evolving linearly with time, while the source positions are assumed to be fixed. The VLBI delay then takes the form:

$$\tau_{ij,a} = \widehat{s_a} \cdot U(\omega)(\overrightarrow{r_i} + \Delta t \overrightarrow{v_i} - \overrightarrow{r_j} - \Delta t \overrightarrow{v_j}) \tag{4}$$

Here $\widehat{s_a}$ is the unit vector pointing at the source labeled by a, and $\overrightarrow{r_i}$, $\overrightarrow{r_j}$ are vectors to the two receivers, and $\overrightarrow{v_i}$, $\overrightarrow{v_j}$ are the velocities of the receivers. The 3×3 matrix $U(\omega)$ accounts for the residual unmodeled rotation between the CRF and the TRF. For simplicity in what follows, I will refer to this rotation as the "EOP". For small rotations we have the approximation:

$$U(\omega) = I + \overrightarrow{\omega} \times \tag{5}$$

here $\overrightarrow{\omega}$ is 3 vector whose components are the rotation about the x, y and z axis in radians. We now proceed to elucidate the symmetries of the delay.

Global translation of positions. The delay is invariant under a simultaneous constant translation of all the stations:

$$\overrightarrow{r_i} \rightarrow \overrightarrow{r_i} + \overrightarrow{\Delta r} \tag{6}$$

Since $\overrightarrow{\Delta r}$ is a 3-vector, this leads to three independent constraints.

Global translation of velocities. The delay is invariant under a simultaneous translation (boost) of the velocities:

$$\overrightarrow{v_i} \rightarrow \overrightarrow{v_i} + \overrightarrow{\Delta v} \tag{7}$$

Since $\overrightarrow{\Delta v}$ is a 3-vector, this leads to three independent constraints.

We now turn our attention to the symmetries associated with the connection of the CRF and TRF.

Constant rotations. There are sets of rotations in VLBI. 1.) We can rotate the CRF; 2.) We can rotate the TRF; 3.) We can change the EOP parameters. In the immediate following we deal with constant rotations, i.e., time invariant. As long as the product of these three rotations is equal to 1, then the delay will be invariant. Physically all we are saying is that a rotation of the CRF and the TRF can be absorbed into a constant change in EOP.

A simultaneous small rotation of all stations in the TRF is specified in terms of the three rotation angles $\overrightarrow{\alpha}$ by:

$$\begin{aligned} \overrightarrow{r_i} &\rightarrow (I + \overrightarrow{\alpha} \times) \overrightarrow{r_i} \\ \overrightarrow{v_i} &\rightarrow (I + \overrightarrow{\alpha} \times) \overrightarrow{v_i} \end{aligned} \tag{8}$$

Similarly, a small rotation of the CRF is given by:

$$\widehat{s_a} \rightarrow (I + \overrightarrow{\beta} \times) \widehat{s_a} \tag{9}$$

while a constant shift in EOP is:

$$\overrightarrow{\omega} \rightarrow \overrightarrow{\omega} + \overrightarrow{\gamma} \tag{10}$$

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Note that if

$$\overrightarrow{\alpha} - \overrightarrow{\beta} + \overrightarrow{\gamma} = 0 \tag{11}$$

then the delay is unchanged up to first order. The minus sign for $\overrightarrow{\beta}$ arises since for any vector $\overrightarrow{q}$ we have

$$\overrightarrow{\beta} \times \widehat{s_a} \cdot \overrightarrow{q} = -\widehat{s_a} \cdot \overrightarrow{\beta} \times \overrightarrow{q}$$

Each of the rotations $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$, and $\overrightarrow{\gamma}$ have 3 parameters, which leads to a total of 9 rotation parameters. The condition that the delay be unchanged given by 11 supplies 3 conditions on the rotations, which reduces the number of independent rotations to 6.

Constant spin. In addition to a constant rotation, one can imagine a rotation where the TRF is rotating with a constant rate of rotation. To distinguish this state from normal rotation, I will call it spin. Such a change can always be absorbed into a linear drift in the EOP. The notion of a spinning TRF is not far fetched. Since the tectonic plates are spinning with respect to each other with a constant angular velocity (constant over the time scales we are interested in), it is impossible to define a frame in which all stations are fixed. The best that you can do is define a frame which is fixed to one of the plates, and spinning with respect to the others.

Under such a "spin transformation", the positions and velocities will change as follows:

$$\begin{aligned} \overrightarrow{r_i} &\rightarrow (I + t\overrightarrow{\alpha} \times)\overrightarrow{r_i} \\ \overrightarrow{v_i} &\rightarrow (I + t\overrightarrow{\alpha} \times)\overrightarrow{v_i} \end{aligned} \tag{12}$$

This form has the disadvantage that the position is now time dependent. If we absorb this time dependence in the velocity, we have:

$$\begin{aligned} \overrightarrow{r_i} &\rightarrow \overrightarrow{r_i} \\ \overrightarrow{v_i} &\rightarrow \overrightarrow{v_i} + \overrightarrow{\alpha} \times \overrightarrow{r_i} + t\overrightarrow{\alpha} \times \overrightarrow{v_i} \\ &\simeq \overrightarrow{v_i} + \overrightarrow{\alpha} \times \overrightarrow{r_i} \end{aligned} \tag{13}$$

In going to the last line I have used the fact that $t\overrightarrow{\alpha} \times \overrightarrow{v_i}$ is very small, and can be ignored. Hence a spin transformation can be used as leaving the positions invariant, but changing the velocity. A constant change in EOP rate is given by:

$$\overrightarrow{\omega} \rightarrow \overrightarrow{\omega} + t\overrightarrow{\theta} \tag{14}$$

Note that if

$$\overrightarrow{\alpha} + \overrightarrow{\theta} = 0$$

the delay will remain unchanged. Each of the 2 angular rates has 3 components, for a total of 6 rotation rate parameters. The condition that the delay be unchanged imposes three conditions on these 6 equations, leaving 3 independent components.

In the above analysis we assumed that the CRF did not spin. If we want to solve for proper motion of the stars, then we need to allow for the CRF to spin. The analysis would be modified

in a way entirely analogous to that of the case of constant rotation, and we would have 9 rotation rate parameters, with 3 conditions on them, leading to a total of 6 independent parameters.

Table 1 below summarizes the invariances of the delay, and what they are associated with.

Table 1. Symmetries of the Delay	
Source/Name	#
Position translation	3
Velocity translation	3
Rotation of CRF,TRF and constant change in EOP	6
Spin of TRF, and EOP slope	3
Total	15

In summary we have 15 transformations which leave the delay invariant. A given VLBI solution is a member of a 15 parameter family of solutions, all which agree equally well with the data (have the same χ^2). If you rely only on VLBI data, there is no way to pick a "preferred solution". You need to invoke other principles or reasons for choosing a particular solution. For example, other techniques make give information which is complementary to VLBI. Mathematically the invariance of the delay leads to a degeneracy of the normal equations–the normal matrix has 0 eigenvectors. The span of the null space is 15 dimensions. Put another way, you can't invert the normal equations as they stand. You need to impose some constraints. The minimum number of constraints is 15. In the following sections we explore various constraints.

5   Old Fashioned VLBI Constraints

In the preceding sections we saw that you needed to impose a minimum of 15 constraints to invert the VLBI normal equations. There are literally an infinite number of ways of choosing such constraints. In this section we describe one such set.

Prior to 1992, the following constraints were imposed on the VLBI solution:

1. A station was held fixed. Both the position and the velocity adjustments were constrained to be 0. This imposed 3 constraints on the position, and 3 on the velocity for a total of 6 constraints. Usually the fixed station was Westford, Massachusetts. This was actually done by "turning off" the adjustments for these parameters. That is, these parameters were removed from the normal equations.

2. The azimuth angle from the fixed station to another station was held fixed. For example, the direction vector from Westford to Richmond, Florida, was constrained to its nominal value. This imposed 2 rotational constraints on the position and 2 on the velocity, for a total of 4 constraints.

3. To prevent rotation about this axis, another rotational constraint is imposed. The up adjustment and velocity of a third station, typically Kaui, were held fixed. This is an additional 2 constraints, 1 each for position and velocity.

4. To fix the orientation of the CRF and TRF two additional sets of constraints were imposed. A) The EOP parameters were not adjusted on some day, for a total of 5 constraints (3 for

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EOP proper, and two for nutation); and B.) The adjustment of the right-ascension of some source, or the average adjustment of some set of sources was constrained to be 0, for one additional constraint.

The total number of constraints is 18, which is larger than the minimal number of constraints, which is 15. The above system actually over-constrain the system. In addition to this, there are several other disadvantages to this system of constraints.

1. The entire nature of the constraints is somewhat ad hoc. To fix the TRF you need to choose some special stations. These stations are typically chosen with some physical considerations. For example, most measurements of Westford indicate that it is stationary, so keeping it fixed seems reasonable. Similarly, Richmond does not appear to be rotating with respect to Westford, so fixing this azimuth also seems reasonable.

2. Although the special stations appear stationary, they may not be. If they have real motion this will map directly into a motion of the whole network.

3. By construction the uncertainties in the position and velocity of the fixed station is 0. Because of the structure of the correlation matrix, this has the unwanted effect that, all other things being equal, stations that are geographically close to the fixed station (e.g., Haystack is close to Westford) will have smaller sigmas while stations that are further from the fixed station will have larger sigmas.

4. There are similar problems with the CRF. By construction, the formal errors for the source whose RA is fixed will vanish. Again, all other things being equal, sources which are close to this will have smaller formal errors while sources which are further away will have larger ones.

5. Yet again, all other things being equal, the EOP formal errors for days which are close in time to the fixed day have smaller formal errors than those which are further away.

6. Fixing EOP implicitly determines the orientation of the CRF. This orientation can only be determined as well as the EOP could be measured. Hence the overall uncertainty of the CRF is effected by the choice of fixed day. Even if the EOP is well determined, there is another artifact.

6   New No Net Constraints

Because of the problems associated with the traditional constraints, I proposed a new series of constraints. These were motivated by the following considerations. Apart from source structure, one expects the relative positions of stars in the CRF to be constant. Furthermore, there is ample evidence that as a good first approximation, stations evolve linearly with time, although there do appear to be systematic seasonal fluctuations in the estimate of station position. Both the TRF and CRF are rigid time invariant models. In contrast, the EOP varies day by day. Furthermore, one does not want the TRF or CRF to be skewed by either a bad choice of reference day, or by unfortunate choices for station and source that appears in the constraint. These lead me to make the following proposals.

1. The orientations of the CRF and the TRF should be constrained, and EOP be determined from them.

2. The TRF and the CRF should agree as closely as possible consistent with the data with the a priori model. That is, the adjustments should be as small as possible.

The advantage of the second principle is that any deviation of the solution from the a priori model is due entirely to the data, and is not due, for example to a poor choice of constraints. The explicitly realization of these principles was done by imposing the following constraints on the VLBI normal equations:

1. The position adjustments for some set of stations is constrained so that the average position adjustment is zero. We refer to this as No Net Translation of the Position (NNTP).

2. The velocity adjustments for some set of stations is constrained so that the average velocity adjustment is zero. We refer to this as No Net Translation of the Velocity (NNTV).

3. The orientation of some set of station positions is constrained so that there is No Net Rotation of the Position (NNRP) of the stations with respect to the a priori model.

4. The velocities of some set of stations is constrained so that the there is No Net Spin of the Velocity with respect to the a priori. Strictly speaking this constraint should be called NNSV, but it is actually called NNRV.

5. The position of some set of sources is constrained so that there is No Net Rotation of the Sources (NNRS) with respect to the a priori.

In the following sections I discuss each of these constraints in more detail, and give their explicit forms.

7   No Net Translation Position Constraints

In order to remove the translational degeneracy, we need to impose constraints on the position. One possibility is to constrain the sum of the position adjustments $\delta \vec{r}_j$ to 0:

$$\sum_j \delta \vec{r}_j = 0 \tag{15}$$

It is worth emphasizing that this constraint makes the VLBI solution be as close as possible to the a priori model. To see this, suppose that we have a solution to the VLBI equations, that is a set of position adjustments, which doesn't neccesarily obey the above constraint. We can obtain another solution by translating all of the adjustments by a constant amount:

$$\delta \vec{r}_j \rightarrow \delta \vec{r}_j + \Delta \vec{r} \tag{16}$$

This set of adjustments is an equally good solution to the normal equations. A natural measure of the distance between the model and the a priori is the sum of squares of the adjustments:

$$D^2 \equiv \sum_j \left(\delta \vec{r}_j + \Delta \vec{r} \right)^2 \tag{17}$$

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We can minimize this adjustment by varying $\Delta \vec{r}$. We find that the D^2 will be a minimum when:

$$\Delta \vec{r} = -\sum_j \delta \vec{r}_j \tag{18}$$

This equation tells us how to adjust the positions so that the VLBI solution is as close as possible to the a priori. Note that if the if the Left Hand Side (LHS) of this equation is 0, then no adjustment is necessary. This condition is precisely the constraint that we have imposed. This sum can be modified in various ways.

There are several variants of the NNTP constraint. For example, the sum could extend over only those stations that are "well determined", or the sum could be replaced by a weighted sum:

$$\sum_j w_j \delta \vec{r}_j = 0 \tag{19}$$

This constraint is called "No Net Translation of Position" because the sum of the position adjustments is zero. This is a vector equation, and imposes three constraints on the position.

An alternate form the NNT Position constraint is also used. To give it's explicit form I need to some notation. Define the site dependent projection operator

$$P_{H_j} = I - \hat{r}_j \otimes \hat{r}_j \tag{20}$$

This acts on an arbitrary vector $\vec{b}$ according to:

$$P_{H_j} \vec{b} = \vec{b} - \left(\vec{b} \cdot \hat{r}_j \right) \hat{r}_j \tag{21}$$

Now $\hat{r}_j$ is in the direction of local up. Let $\hat{e}_j$ and $\hat{n}_j$ be the local East and North unit vectors. It is straightforward to verify that:

$$\begin{aligned}
P_{H_j} \hat{r}_j &= 0 \\
P_{H_j} \hat{e}_j &= \hat{e}_j \\
P_{H_j} \hat{n}_j &= \hat{n}_j
\end{aligned} \tag{22}$$

Also, for any vector $\vec{b}$ we have

$$\hat{r}_j P_{H_j} \vec{b} = 0$$

Hence P_{H_j} projects a vector onto the horizontal components. With this background, an alternate constraint is:

$$\sum_j P_{H_j} \delta \vec{r}_j = 0$$

This constrains the average horizontal projection to be 0. For the position there is no strong argument for using this particular constraint, although there is for the analogous case of the velocities.

8 No Net Translation Velocity Constraints

In order to remove the translational degeneracy for the velocities, we need to impose constraints on the velocity. One possibility is to constrain the sum of the velocity adjustments $\delta \overrightarrow{v}_j$ to 0:

$$\sum_j \delta \overrightarrow{v}_j = 0 \tag{23}$$

As with the position constraint, this sum can be modified in various ways. For example, the sum could extend over only those stations where the velocity is "well determined", or the sum could be replaced by a weighted sum:

$$\sum_j w_j \delta \overrightarrow{v}_j = 0 \tag{24}$$

This constraint is called "No Net Velocity Translation" because the sum of the position adjustments is zero. This is a vector equation, and imposes three constraints. As with the No Net Position Translation Constraint, this constraint makes the solution as close as possible to the a priori model consistent with the data. The argument of the previous section goes through essentially intact, *mutatis mutandis*, and won't be repeated here.

An alternate form the NNT velocity constraint is also used:

$$\sum_j P_{H_j} \delta \overrightarrow{v}_j = 0$$

This constrains the average horizontal projection of the velocity to 0. One motivation for this is the a priori models may give no information about the vertical components. For example, the plat models make predictions about the horizontal components of the velocities, but make no predictions about the vertical. If we constrain the velocities to be as close as possible to the a priori model, we are implicitly constraining the average vertical velocity to be 0. We may or may not want to do this, but we should be aware of what we are doing.

9 No Net Rotation of Position

To fix the orientation of the TRF I need to impose a constraint on the orientation. The constraint I use is:

$$\sum_j \frac{1}{r_{0,j}^2} \overrightarrow{r}_{0,j} \times \delta \overrightarrow{r}_j = 0 \tag{25}$$

The constraint can be modified to be a weighted sum, or to include only some stations. Here $\overrightarrow{r}_{0,j}$ is the a priori station position. It is interesting to note that a given term in the sum

$$\frac{1}{r_{0,j}^2} \overrightarrow{r}_{0,j} \times \delta \overrightarrow{r}_j$$

is exactly the rotation that site would have to go through to be aligned with the a priori model. To see this, note that under this rotation, the total vector, a priori plus adjustment, will undergo

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the following transformation:

$$(\overrightarrow{r}_{0,j} + \delta\overrightarrow{r}_j) \;\; \rightarrow \;\; \left(I + \left(\frac{1}{r_{0,j}^2}\overrightarrow{r}_{0,j} \times \delta\overrightarrow{r_j} \right) \times \right) (\overrightarrow{r}_{0,j} + \delta\overrightarrow{r}_j) \tag{26}$$

$$= \;\; \overrightarrow{r}_{0,j} + (\delta\overrightarrow{r_j} \cdot \hat{r}_0)\,\hat{r}_0$$

The second term on the RHS is the vertical part of the adjustment, which can't be eliminated by a rotation. The interpretation of the constraint is clear: The sum of the rotations between the solution and the a priori is 0, or, in words, there are No Net Rotations.

There is an alternate derivation of this constraint which demonstrates that this makes the solution be as close as possible to the a priori. The argument follows that for the translational constraints. Suppose we have some solution to the VLBI normal equations. We can obtain another solution by rotation:

$$(\overrightarrow{r}_{0,j} + \delta\overrightarrow{r}_j) \;\; \rightarrow \;\; (I + \overrightarrow{\alpha}\times)\,(\overrightarrow{r}_{0,j} + \delta\overrightarrow{r}_j)$$

$$= \;\; \overrightarrow{r}_{0,j} + \delta\overrightarrow{r}_j + \overrightarrow{\alpha} \times \overrightarrow{r}_{0,j} + \overrightarrow{\alpha} \times \delta\overrightarrow{r_j}$$

$$\simeq \;\; \overrightarrow{r}_{0,j} + \delta\overrightarrow{r}_j + \overrightarrow{\alpha} \times \overrightarrow{r}_{0,j}$$

In going from the second to the third line I have used the fact that last term is much small compared to the other terms. The distance squared of this new solution from the a priori is:

$$D^2 \equiv \sum_j \left(\delta\overrightarrow{r}_j + \overrightarrow{\alpha} \times \overrightarrow{r}_{0,j} \right)^2$$

This can be minimized with respect to $\overrightarrow{\alpha}$ to obtain:

$$\left(\sum_j P_{H_j} \right) \overrightarrow{\alpha} = -\sum_j \frac{1}{r_{0,j}^2}\overrightarrow{r}_{0,j} \times \delta\overrightarrow{r}_j$$

where is the horizontal projection operator introduced earlier. This equation tells us how to find the rotation we need to align the VLBI solution as closely as possible with the a priori model. Note that if the Right Hand Side of this is 0, then the required rotation is 0. The condition that the RHS be 0 is precisely our constraint.

10   No Net Velocity Rotation

To remove the degeneracy associated with a spin of the TRF I need to impose a constraint on the orientation. The constraint I use is:

$$\sum_j \frac{1}{r_{0,j}^2}\overrightarrow{r}_{0,j} \times \delta\overrightarrow{v}_j = 0 \tag{27}$$

Here $\overrightarrow{r}_{0,j}$ is the a priori station position. It is interesting to note that a given term in the sum

$$\frac{1}{r_{0,j}^2}\overrightarrow{r}_{0,j} \times \delta\overrightarrow{v}_j$$

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is exactly the spin required to reduce the horizontal velocity of the site with respect to the a priori model to 0. Under a spin transformation the total velocity will transform according to:

$$
\begin{aligned}
(\overrightarrow{v}_{0,j} + \delta \overrightarrow{v}_j) \quad &\rightarrow \quad (\overrightarrow{v}_{0,j} + \delta \overrightarrow{v}_j) + \left(\frac{1}{r_{0,j}^2} \overrightarrow{r}_{0,j} \times \delta \overrightarrow{v_j} \right) \overrightarrow{r}_{0,j} \qquad (28) \\
&= \quad \overrightarrow{v}_{0,j} + (\delta \overrightarrow{v_j} \cdot \widehat{r}_0) \, \widehat{r}_0
\end{aligned}
$$

The second term on the RHS is the vertical part of the velocity adjustment, which can't be eliminated by a spin transformation. The interpretation of the constraint is clear: The sum of the rotations between the solution and the a priori is 0, or, in words, there is No Net Spin. For historical reasons this is actually (incorrectly) called No Net Rotation of the Velocities.

The argument of the proceeding section goes through essentially intact. This constraint on the velocities aligns the a priori velocities and the VLBI solutions as closely as possible.

11   No Net Source Rotation Position

To remove the rotational degeneracy in the CRF, I need to impose a constraint on the orientation. The constraint I use is:

$$
\sum_a \widehat{s}_{0,a} \times \delta \widehat{s}_a = 0 \qquad (29)
$$

Here $\widehat{s}_{0,a}$ is the a priori source position. As in the previous cases, each term in the some is the rotation that particular source would have to undergo to be aligned with the a priori source model. The sum of the rotations between the solution and the a priori is 0, or, in words, there is No Net Rotation of the Sources. As in the proceeding sections, one can show that this constraint aligns the VLBI solution as closely as possible with the a priori model.

In contrast to the previous cases, it is not obvious that the constraint equation produces three independent constraints. I will now show that this is the case. The unit vector in the direction of a source is given by:

$$
s_a = (\cos Dec_a \cos RA_a, \cos Dec_a \sin RA_a, \sin Dec_a)
$$

Suppose the RA and the Dec of the source are given by

$$
\begin{aligned}
RA_a &= RA_{a,0} + \delta RA_a \\
Dec_a &= Dec_{a,0} + \delta Dec_a
\end{aligned}
$$

then the unit vector takes the form:

$$
\widehat{s}_a = \widehat{s}_{a,0} + \delta RA_a \frac{\partial \widehat{s}_a}{\partial RA_a} + \delta Dec_a \frac{\partial \widehat{s}_a}{\partial Dec_a}
$$

It is straightforward to evaluate the derivatives, and we find:

$$
\begin{aligned}
\frac{\partial \widehat{s}_a}{\partial RA_a} &= (-\cos Dec_{a,0} \sin RA_{a,0}, \cos Dec_{a,0} \cos RA_{a,0}, 0) \\
\frac{\partial \widehat{s}_a}{\partial RA_a} &= (-\sin Dec_{a,0} \cos RA_a, -\sin Dec_{a,0} \sin RA_a, \cos Dec_{a,0})
\end{aligned}
$$

The constraint equation becomes the three equations:

$$\sum_a \left(\delta Dec_a \sin RA_{a,0} - \delta RA_a \sin Dec_{a,0} \cos Dec_{a,0} \cos RA_a\right) = 0$$

$$\sum_a \left(\delta Dec_a \cos RA_{a,0} - \delta RA_a \sin Dec_{a,0} \cos Dec_{a,0} \sin RA_a\right) = 0$$

$$\sum_a \delta Dec_a \cos^2 Dec_{a,0} = 0$$

It is this latter form that is actually used to constrain the solution.

12   Conclusion

In the preceding sections we have discussed the symmetries of the VLBI delay. To each symmetry there is an associated degeneracy of the normal equations. To remove these degeneracies you need to apply constraints. We discussed two kinds of constraints: Those in use prior to 1992, and those currently in use. The older constraints were of an ad hoc nature, with no overall defining philosophy. The newer constraints involve picking a set of stations and demanding that the these stations be as close as possible to the a priori model as is consistent with the data.

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